The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane.

Part Second: Complete Form-System of Invariants.*

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§1.—Invariants of the Binary Quintic Form.

1. Three invariants of the quintic as given by Salmon† are:

$$\left.\begin{aligned} P^2 &= \Pi\,(12)^2, \\ A &= \Sigma\,(12)^4\,(34)^2\,(35)^2\,(45)^2, \\ C &= P^2\,\Sigma\,(12)^{-4}\,(34)^{-2}\,(35)^{-2}\,(45)^{-2}, \end{aligned}\right\} \tag{1}$$

where (ij) means the root difference

$$(\alpha_i - \alpha_j), \qquad (i, j = 1 \ldots . 5).$$

2. The ratio $A : P$ written in cross-ratio form is

$$\left.\begin{aligned} A : P = {} & [3214][3415][3512] + [4123][4325][4521] \\ & + [5132][5234][5431] + [5213][5314][5412] \\ & - [2314][2415][2513] - [3124][3425][3521] \\ & - [4132][4235][4531] - [5142][5243][5341] \\ & - [5123][5324][5421] - [4213][4315][4512] \end{aligned}\right\} \tag{2}$$

in which

$$[ijkl] = \frac{(\alpha_i - \alpha_k)(\alpha_j - \alpha_l)}{(\alpha_j - \alpha_k)(\alpha_i - \alpha_l)}, \qquad (i, j, k, l = 1 \ldots . 5).$$

If we put in (2) $\qquad \lambda_1 = [4235],$

* The first part of this memoir is to be found in the *American Journal of Mathematics*, vol. XXII, pp. 343–388, 1900. It will be referred to here simply as Part First.

† "Modern Higher Algebra," third edition, §241.

and let $\lambda_2 \ldots . \lambda_5$ be derived from λ_1 by successive application of the cyclic permutation (12345) to the indices, and make use of the relations*

$$\lambda_3 = \frac{1-\lambda_1}{1-\lambda_1\lambda_2}, \quad \lambda_4 = 1-\lambda_1\lambda_2, \quad \lambda_5 = \frac{1-\lambda_2}{1-\lambda_1\lambda_2}, \tag{3}$$

we get

$$\begin{aligned} A:P = -\{(1-\lambda_1)^2[1+\lambda_2^2(1-\lambda_1)^2+\lambda_1^2\lambda_2^4] + (1-\lambda_2)^2[1+\lambda_1^2(1-\lambda_2)^2+\lambda_1^4\lambda_2^2] \\ + (1-\lambda_1\lambda_2)^2[\lambda_1^2+\lambda_2^2+(1-\lambda_1\lambda_2)^2+(1-\lambda_1)^2(1-\lambda_2)^2]\} \\ :\{\lambda_1\lambda_2(1-\lambda_1)(1-\lambda_2)(1-\lambda_1\lambda_2)\}. \end{aligned} \tag{4}$$

3. We now identify the roots α_i with the variables† ν_i in the following order:

$$\left.\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \nu_5 & \nu_2 & \nu_3 & \nu_1 & \nu_4 \end{matrix}\right\} \tag{5}$$

and obtain the relation

$$\lambda_1 = \rho, \quad \lambda_2 = \frac{\sigma-1}{\sigma-\rho}. \tag{6}$$

Substituting (6) in (4) and changing to homogeneous coordinates

$$\rho = \frac{z_1}{z_3}, \quad \sigma = \frac{z_2}{z_3},$$

*M. J. M. Hill, "The Anharmonic Ratios of the Roots of a Quintic" (Proceedings of the London Mathematical Society, vol. XIV, p. 182).

† E. H. Moore, "The Cross-Ratio Group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions" (*American Journal of Mathematics*, vol. XXII, p. 280, 1900). For the case $n=5$, I use the variables $\nu_1 \ldots . \nu_5$, and the fundamental system of cross-ratios,

$$\rho \equiv \rho_4 = [\nu_1\nu_2\nu_3\nu_4] \text{ and } \sigma \equiv \rho_5 = [\nu_1\nu_2\nu_3\nu_5].$$

Whence, ρ_1, ρ_2, ρ_3 give the special values ∞, 0, 1.

With this notation, for example, the transformation T, corresponding to the substitution on the indices (15)(34), is derived as follows:

$$\rho' = \rho_{5243} = \frac{(\rho_5-\rho_4)(\rho_2-\rho_3)}{(\rho_2-\rho_4)(\rho_5-\rho_3)} = \frac{\rho-\sigma}{\rho(1-\sigma)},$$

$$\sigma' = \rho_{5241} = \frac{(\rho_5-\rho_4)(\rho_2-\rho_1)}{(\rho_2-\rho_4)(\rho_5-\rho_1)} = \frac{\rho-\sigma}{\rho},$$

which, in homogeneous coordinates, becomes

$$T; \quad z_1' : z_2' : z_3' = z_3(z_1-z_2) : (z_1-z_2)(z_3-z_2) : z_1(z_3-z_2).$$

See Part First, Arts. 4, 28 and 58.

there results

$$\left.\begin{aligned} A : P = -\{&\Sigma z_1^2 (z_2 - z_3)^4 + \Sigma z_1^4 (z_2 - z_3)^2 + \Sigma z_1^2 z_2^2 (z_1 - z_2)^2 \\ &+ (z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 - z_3)^2\} : \{z_1 z_2 z_3 (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)\}. \end{aligned}\right\} \quad (7)$$

In a similar manner,

$$\left.\begin{aligned} C : P^3 = -\{&\Sigma z_1^4 z_2^4 (z_1 - z_2)^2 (z_1 - z_3)^4 (z_2 - z_3^4) + \Sigma z_1^2 z_2^4 z_3^4 (z_1 - z_2)^4 (z_1 - z_3)^4 \\ &+ \Sigma z_1^2 z_2^2 z_3^4 (z_1 - z_2)^2 (z_1 - z_3)^4 (z_2 - z_3)^4 + z_1^4 z_2^4 z_3^4 (z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 - z_3)^2\} \\ &: \{z_1^3 z_2^3 z_3^3 (z_1 - z_2)^3 (z_1 - z_3)^3 (z_2 - z_3)^3\}. \end{aligned}\right\} \quad (8)$$

4. Applying to (7) and (8) the transformation

$$z_1 : z_2 : z_3 = y_1 - y_4 : y_2 - y_4 : y_3 - y_4, \quad (9)$$

we find the functions proportional to A, P^2, C;

$$\left.\begin{aligned} \delta_1 A &= \Sigma (y_1 - y_2)^2 (y_3 - y_4)^4 + \Sigma (y_1 - y_2)^2 (y_1 - y_3)^2 (y_2 - y_3)^2, \\ \delta_2 P^2 &= (y_1 - y_2)^2 (y_1 - y_3)^2 (y_1 - y_4)^2 (y_2 - y_3)^2 (y_2 - y_4)^2 (y_3 - y_4)^2, \\ \delta_3 C &= \Sigma (y_1 - y_2)^2 (y_1 - y_3)^4 (y_1 - y_4)^4 (y_2 - y_3)^4 (y_2 - y_4)^4 \\ &\qquad + P^2 \Sigma (y_1 - y_4)^2 (y_2 - y_4)^2 (y_3 - y_4)^2. \end{aligned}\right\} \quad (10)$$

5. It is desirable to evaluate A, P^2 and C in terms of the elementary symmetric functions:

$$-p_1 = \Sigma y_i, \quad p_2 = \Sigma y_i y_j, \quad -p_3 = \Sigma y_i y_j y_k, \quad p_4 = y_1 y_2 y_3 y_4, \quad (11)$$
$$(i, j, k = 1 \ \ldots . \ 4).$$

The results for A and P^2 are easily found:

$$\left.\begin{aligned} \delta_1 A &= 3^2 p_3^2 + 2 \cdot 3 p_2^3 + 2^3 \cdot 5 p_2 p_4 + \ldots . \text{ terms containing } p_1 \text{ as a factor,} \\ \delta_2 P^2 &= 2^8 p_4^3 - 3^3 p_3^4 - 2^2 p_2^3 p_3^2 - 2^7 p_2^2 p_4^2 + 2^4 p_2^4 p_4 + 2^4 \cdot 3^2 p_2 p_3^2 p_4 \\ &\qquad + \ldots . \text{ terms containing } p_1 \text{ as a factor.} \end{aligned}\right\} \quad (12)$$

6. For the first part of C, whose weight is 18, we have

$$\left.\begin{aligned} \Sigma (y_1 - y_2)^2 (y_1 - y_3)^4 (y_1 - y_4)^4 (y_2 - y_4)^4 (y_3 - y_4)^4 &\equiv a_1 p_2^9 + a_2 p_3^6 \\ + a_3 p_3^2 p_4^3 + a_4 p_2^6 p_3^2 + a_5 p_2^3 p_3^4 + a_6 p_2^7 p_4 &+ a_7 p_2^5 p_4^2 + a_8 p_2^3 p_4^3 \\ + a_9 p_2 p_4^4 + a_{10} p_2 p_3^4 p_4 + a_{11} p_2^4 p_3^2 p_4 &+ a_{12} p_2^2 p_3^2 p_4^2 \\ + 41 \text{ terms containing } p_1 \text{ as a factor.}& \end{aligned}\right\} \quad (13)$$

7. In (13) put

$$y_1 = -y_3 = 1, \quad y_2 = y_4 = 0,$$

Whence by (11), $p_1 = p_3 = p_4 = 0, \quad p_2 = -1.$

From which $a_1 = -2^2.$

To determine a_2, a_4, a_5, put in succession in (13),

$$y_1, y_2, y_3, y_4 = 0, 2, -1, -1; \; 0, -3, 2, 1; \; 0, 4, -3, -1.$$

whence

$$p_1, p_2, p_3, p_4 = 0, -3, -2, 0; \; 0, -7, 6, 0; \; 0, -13, -12, 0.$$

These substitutions lead to the conditions,

1) $2^4 a_2 + 3^6 a_4 - 2^2 \cdot 3^3 a_5 = -2 \cdot 3^8,$
2) $2^4 \cdot 3^6 a_2 + 3^2 \cdot 7^6 a_4 - 2^2 \cdot 3^4 \cdot 7^3 a_5 = 2^{11} \cdot 5^5 + 3^4 \cdot 5^5 + 2^{11} \cdot 3^4 \cdot 13 - 7^9,$
3) $2^{10} \cdot 3^6 a_2 + 2^2 \cdot 3^2 \cdot \overline{13}^6 a_4 - 2^6 \cdot 3^4 \cdot \overline{13}^3 \, a_5 = 3^4 5^5 \cdot 7^4 + 2^{11} \cdot 7^4 \cdot 17 + 2^{11} \cdot 3^4 \cdot 5^6 - \overline{13}^9,$

from which

$$a_2 = -2 \cdot 3^5, \quad a_4 = -2 \cdot 3 \cdot 11, \quad a_5 = -2^2 \cdot 3^2 \cdot 11.$$

8. To find a_6, a_7, a_8, a_9, put in succession,

$$y_1, y_2, y_3, y_4 = 1, -1, 1, -1; \; 1, -1, 2, -2; \; 1, -1, 3, -3; \; 2, -2, 3, -3$$

Whence,

$$p_1, p_2, p_3, p_4 = 0, -2, 0, 1; \; 0, -5, 0, 4; \; 0, -10, 0, 9; \; 0, -13, 0, 36.$$

Then from the equations:

1) $2^6 a_6 + 2^4 a_7 + 2^2 a_8 + a_9 = 2^{10},$
2) $5^6 a_6 + 2^2 \cdot 5^4 a_7 + 2^4 \cdot 5^2 a_8 + 2^6 \cdot a_9 = -2^6 \cdot 11 \cdot 29 \cdot 113,$
3) $2^6 \cdot 5^6 a_6 + 2^4 \cdot 3^2 \cdot 5^4 a_7 + 2^2 \; 3^4 \cdot 5^2 a_8 + 3^6 a_9 = -2^{10} \cdot 83,687,$
4) $\overline{13}^6 a_6 + 2^2 \cdot 3^2 \cdot \overline{13}^4 \cdot a_7 + 2^4 \cdot 3^4 \cdot \overline{13}^2 a_8 + 2^6 \cdot 3^6 \cdot a_9 = -2^6 \cdot 7,238,879,$

we get,

$$a_6 = 2^6, \quad a_7 = -2^7 \cdot 11, \quad a_8 = -2^{10} \cdot 7, \quad a_9 = 2^{10} \cdot 47.$$

9. To find a_3, a_{10}, a_{11}, a_{12}, put in succession,

$$y_1, y_2, y_3, y_4 = 1, 1, 1, -3; \; 1, 1, 2, -4; \; 2, 2, -3, -1; \; 1, 1, 4, -6,$$

which give,

$$p_1, p_2, p_3, p_4 = 0, -6, 8, -3; \; 0, -11, 18, -8; \; 0, -9, 4, 12; \; 0, -27, 50, -24.$$

Then from the equations:

$$\begin{array}{lllll}
1) & 3a_3 - 2^7 a_{10} & + 2^4\cdot 3^3 a_{11} & - 2^2\cdot 3^2 a_{12} & = 2^7\cdot 3^3\cdot 7\cdot 11, \\
2) & 2^6 a_3 - 2^2\cdot 3^4\cdot 11 a_{10} & + \overline{11}^4 a_{11} & - 2^3\cdot \overline{11}^2 a_{12} & = 2^3\cdot 1,339,307, \\
3) & 2^4 a_3 - 2^4 a_{10} & + 3^6 a_{11} & + 2^2\cdot 3^2 a_{12} & = 2^3\cdot 3^2\cdot 11\cdot 617, \\
4) & 2^6 a_3 - 2^2\cdot 3\cdot 5^4 a_{10} & + 3^{10} a_{11} & - 2^3\cdot 3^5 a_{12} & = 2^3\cdot 3^2\cdot 7\cdot 11\cdot 9,767,
\end{array}$$

we find,

$$a_3 = -2^7\cdot 3^2\cdot 23, \quad a_{10} = 2^4\cdot 3^3, \quad a_{11} = 2^3\cdot 131, \quad a_{12} = 2^5\cdot 3^2\cdot 5.$$

Thus we have found:

$$\left.\begin{array}{l}
\Sigma (y_1 - y_2)^2 (y_1 - y_3)^4 (y_1 - y_4)^4 (y_2 - y_3)^4) (y_2 - y_4)^4 \equiv - 2^2 p_2^9 - 2\cdot 3^5 p_3^6 \\
\quad - 2^7\cdot 3^2\cdot 23 p_3^2 p_4^3 - 2\cdot 3\cdot 11 p_2^6 p_3^2 - 2^2\cdot 3^2\cdot 11 p_2^3 p_3^4 + 2^6 p_2^7 p_4 - 2^7\cdot 11 p_2^5 p_4^2 \\
\quad - 2^{10}\cdot 7 p_2^3 p_4^3 + 2^{10}\cdot 47 p_2 p_4^4 + 2^4\cdot 3^3 p_2 p_3^4 p_4 + 2^3\cdot 131 p_2^4 p_3^2 p_4 + 2^5\cdot 3^2\cdot 5 p_2^2 p_3^2 p_4^2 \\
\quad + \ldots \text{ terms containing } p_1 \text{ as a factor.}
\end{array}\right\} \quad (14)$$

10. If the second part of C be evaluated in a similar manner and the result combined with (14), we have finally,

$$\left.\begin{array}{l}
\delta_3 C = - 2^2 p_2^9 - 2\cdot 3^3\cdot 23 p_3^6 - 2.\,17 p_2^6 p_3^2 - 2^7\cdot 151 p_3^2 p_4^3 \\
\quad - 2^2\cdot 73 p_2^3 p_3^4 - 2^6 p_2^7 p_4 + 2^7 p_2^5 p_4^2 - 2^{10}\cdot 13 p_2^3 p_4^3 \\
\quad + 2^{10}\cdot 5\cdot 11 p_2 p_4^4 + 2^4\cdot 3^2\cdot 5^2 p_2 p_3^4 p_4 + 2^3\cdot 3^3 p_2^4 p_3^2 p_4 + 2^5\cdot 7\cdot 11 p_2^2 p_3^2 p_4^2 \\
\quad + \ldots \text{ terms containing } p_1 \text{ as a factor.}
\end{array}\right\} \quad (15)$$

§2.—Invariants of the Subgroup $G_{24}^{(1)}$.

Critical and Non-Critical Points. Arts. 11–13.

11. One of the points

$$1i \qquad\qquad (i = 2 \ldots 5)$$

is *non-critical* in the following cases:

(a). In general, for all the linear transformations of G_{120} which form the subgroup $G_{24}^{(1)}$ and which *permute* these four points among themselves in $4\,!$ ways.

(b). In particular, for a certain dihedron subgroup of $G_{24}^{(1)}$, which leaves the corresponding *point fixed.* Thus the point $1i$ is fixed under

$$G_6^{1i} \sim \{jkl\} \text{ all}, \qquad (i, j, k, l = 2 \ldots 5).$$

(c). For such quadratic transformations as leave the *point fixed*, thus $1i$ is fixed under each of the 6 quadratic transformations of the set [Part First, Art. 10],

$$D_{1i}^{-1} \sim \{jkl\} \text{ all } (1i).$$

These belong to the set S_{1i}^{-1}, where

$$S_{1i} \sim \{2345\} \text{ all } (1i) \qquad \text{[Part First, Art. 15]}$$

is the complete set of transformations through any one of which $G_{24}^{(1)}$ is transformed into $G_{24}^{(i)}$.

12. Under *all other quadratic transformations* the point $1i$ is *critical;* that is, it must be regarded as a pencil of directions which goes into a range of points on one of the fundamental sides. [Part First, Art. 12.] These transformations consist of

(a) the remaining 18 in the set S_{1i}^{-1}.

(b) all of the sets, S_{1j}^{-1}, $(j \neq i = 2 \ldots . 5)$.

In all, $3 \cdot 24 + 18 = 90$.

13. *A pencil,* $1i$, *is invariant* under such transformations as permute among themselves its infinity of direction tangents. Evidently this will happen if, and only if, the *corresponding point is fixed.*

As just shown, the point $1i$ is fixed under the *linear* transformations G_6^{1i} and under the *quadratic* transformations of the set D_{1i}^{-1}, and hence the pencil $1i$ is invariant under the linear subgroup,

$$G_6^{1i} \sim \{jkl\} \text{ all},$$

and under the *quadratic subgroup,*

$$G_{12}^{1i} \sim \{jkl\} \text{ all } \{1i\}.*$$

This is in agreement with Part First, Art. 33, where the 4 pencils and 6 sides were found to form a system of 10 conjugate elements under G_{120}.

The linear subgroup G_6^{1i}, which plays an important rôle in the sequel, is also a subgroup of the quadratic group $G_{24}^{(i)}$, its transformations being the *only linear* transformations in that subgroup.

* See foot-note to Part First, Art. 10.

The Complete Form-System for $G_{24}^{(1)}$. Arts. 14–17.

14. The linear subgroup $G_{24}^{(1)}$ is projectively connected with Klein's collineation group $G_{4!}$ by the transformation*

$$\left.\begin{array}{ll}\text{direct;} & z_1 : z_2 : z_3 = x_2 + x_3 : x_3 + x_1 : x_1 + x_2, \\ \text{inverse;} & x_1 : x_2 : x_3 = -z_1 + z_2 + z_3 : z_1 - z_2 + z_3 : z_1 + z_2 - z_3,\end{array}\right\} \qquad (1)$$

where $x_1 \ldots . x_3$ are homogeneous point coordinates; or by the transformation†

$$\left.\begin{array}{ll}\text{direct;} & z_1 : z_2 : z_3 = y_1 - y_4 : y_2 - y_4 : y_3 - y_4, \\ \text{inverse;} & y_1 : y_2 : y_3 : y_4 = 3z_1 - z_2 - z_3 : -z_1 + 3z_2 - z_3 \\ & \qquad : -z_1 - z_2 + 3z_3 : -z_1 - z_2 - z_3,\end{array}\right\} \qquad (2)$$

where $y_1 \ldots . y_4$ are supernumerary homogeneous point coordinates.

15. The complete form-system of invariants of Klein's collineation group G_4 consists of the elementary symmetric functions

$$\Sigma y_i y_j, \quad \Sigma y_i y_j y_k, \quad \Pi y_i, \qquad (i, j, k = 1 \ldots . 4) \quad (3)$$

with the identical relation

$$\Sigma y_i = 0.$$

Hence, the complete form-system of invariants of $G_{24}^{(1)}$ will be derived from that of $G_{4!}$ by applying the transformation (2) to the forms (3). The results are:

$$\left.\begin{array}{rl} p_2 = \Sigma y_i y_j & = -6\Sigma z_1^2 + 4\Sigma z_1 z_2 \\ -p_3 = \Sigma y_i y_j y_k = & 8\Sigma z_1^3 - 8\Sigma z_1^2 z_2 + 16 z_1 z_2 z_3, \\ p_4 = \Pi y_i & = -3\Sigma z_1^4 + 4 z_1^3 z_2 - 20\Sigma z_1^2 z_2 z_3 + 14\Sigma z_1^2 z_2^2,\end{array}\right\} \qquad (4)$$

where again $\Sigma y_i = 0$ identically in the z's.

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*Part First, (5), Art. 7.

†For the case $G'_{4!}$, I introduce supernumerary coordinates conveniently in the form

$$y_1 : y_2 : y_3 : y_4 = -x_1 + x_2 + x_3 : x_1 - x_2 + x_3 : x_1 + x_2 - x_3 : -x \; - x_2 - x_3,$$

from which, in combination with (1), we derive (2). It thus appears that

$$\Sigma y_i = 0$$

identically in the z's as well as in the x's. I thus have the y's related to the z's of $G_{24}^{(1)}$ just as Professor Moore has related the y's to the x's of $G_{4!}$. See his paper, "Concerning Klein's Group of $(n+1)!$ n-ary Collineations" (*American Journal of Mathematics*, vol. XXII, pp. 336–342, 1900).

In these forms put

$$\Sigma z_1 = p, \quad \Sigma z_1 z_2 = q, \quad z_1 z_2 z_3 = r, \tag{5}$$

and they become

$$\left.\begin{aligned} p_2 &= -2\,[3p^2 - 2^3 q] \\ -p_3 &= \quad 2^3\,[\,p^3 - 2^2 pq + 2^3 r], \\ p_4 &= -p\,[3p^3 - 2^4 pq + 2^6 r]. \end{aligned}\right\} \tag{6}$$

16. The forms A, P^2, C, initially given as functions of the roots of the quintic, have been interpreted (a) as functions of the homogeneous coordinates of the transformations of the cross-ratio group G_{120}, and hence of the subgroup $G_{24}^{(1)}$, by the agreement (5), Art. 3; (b) as functions of the supernumerary homogeneous coordinates of the transformations of $G_{4!}$ by virtue of the substitutions (9), Art. 4, since this is the same as (2), Art. 14; and (c) as rational integral functions of p_1, p_2, p_3, p_4 by (12), Art. 5, and (15), Art. 10.

Hence, by Art. 15, they are *absolute invariants* of $G_{24}^{(1)}$. They may be further simplified by substituting in them the value of p_2, p_3, p_4 from (6), Art. 15, and remembering that $p_1 = 0$, thus

$$\delta_1 A = 2p^2q^2 - 2\cdot 3\,(p^3 r + q^3) + 19pqr - 3^2 r^2, \tag{7}$$

$$\delta_2 P^2 = p^2q^2r^2 - 2^2\,(p^3r^3 + q^3r^2) + 2\cdot 3^2 pqr^3 - 3^3 r^4, \tag{8}$$

$$\left.\begin{aligned} \delta_3 C = {} & 2^2\cdot 5^2 p^4q^4r^2 - 2\cdot 3^3\cdot 23 r^6 + 2^4\cdot 5\cdot 7\, p^3q^3r^3 - 2\cdot 5^2\cdot 43 p^2q^2r^4 \\ & + 2\cdot 3^2\cdot 157 pqr^5 - 2\cdot 143\,(p^2q^5r^2 + p^5q^2r^3) - 2\cdot 17\,(p^6r^4 + q^6r^2) \\ & - 2^2\cdot 73\,(p^3r^5 + q^3r^4) + (p^2q^8 + p^8q^2r^2) + 2\cdot 5\cdot 53(p^4qr^4 + pq^4r^3) \\ & + 2\cdot 5^2\,(p^7qr^3 + pq^7r) - 2^2\,(p^9r^3 + q^9) - 2^2\cdot 3\,(p^6q^3r^2 + p^3q^6r). \end{aligned}\right\} \tag{9}$$

17. The form P^2 is of special interest later.

In terms of $y_1 \ldots\ldots y_4$ for the group $G_{4!}$, we have by (10), Art. 4,

$$P_y = \Pi\,(y_1 - y_2) \equiv \sqrt{\Delta}, \tag{10}$$

and in terms of $z_1 \ldots\ldots z_3$ for the group $G_{24}^{(1)}$, by (7), Art. 3,

$$P_z = z_1 z_2 z_3\,(z_1 - z_2)(z_1 - z_3)(z_2 - z_3), \tag{11}$$

in which latter the factors on the right give the six sides of the quadrangle [Fig. II, Part First].

By well-known principles,* the linear homogeneous substitution group $G_{n!}$, which is isomorphic with the symmetric permutation group on n letters, has one, and only one, *fundamental relative invariant*, namely, $\sqrt{\Delta}$, since (except for $n=4$) it has one, and only one, self-conjugate subgroup $G_{\frac{n!}{2}}$.

Since the index of this subgroup is 2, it follows that the only relative invariants under $G_{n!}$ $(n \neq 4)$ must throw off the factor (-1).

This conclusion, however, holds also for $n=4$, since $G_{4!}$ can be generated by transformations *all of period* 2,

$$K', \; L', \; M', \qquad \text{[Part First, (3), Art. 6.]}$$

so that if any primitive root of unity higher than the second could be thrown off, it would have to be built out of the factors $(+1)$ and (-1), which is impossible.

Hence, P_y is the only fundamental relative invariant under $G_{4!}$, from which it follows that P_z is the only such form under $G_{24}^{(1)}$.

Therefore, all relative invariant forms under $G_{24}^{(1)}$ must be of the form

$$P^{2\kappa+1} \cdot f(p_2, p_3, p_4), \tag{12}$$

where f is a rational integral function.

§3.—Characteristics of Invariants Under G_{120}.

Fundamental Notions and Definitions. Arts. 18–21.

18. Since a quadratic transformation, when applied to any function of the z's, must double its degree, it follows that no such function can be *invariant in the ordinary sense under* G_{120}.

The only invariant form possible is a *rational fraction* such that, under any quadratic transformation of the group, a common factor (a function of the z's) is thrown off in numerator and denominator.

Evidently, the degree of the numerator must be the same as that of the denominator and equal to that of the factor thrown off.

19. Since a linear transformation does not change the degree of the function operated upon, it follows that numerator and denominator of an invariant fraction must each be an absolute or relative invariant under $G_{24}^{(1)}$.

* Weber, "Lehrbuch der Algebra," vol. II, p. 161–164.

Such a fraction cannot have both terms relative invariants under $G_{24}^{(1)}$, for [Art. 17, (12)] it would have the form

$$\frac{P^{2\kappa_1+1}\cdot\phi_1(p_2, p_3, p_4)}{P^{2\kappa_2+1}\cdot\phi_2(p_2, p_3, p_4)},$$

which reduces, according as $\kappa_1 \gtrless \kappa_2$, to

$$\text{(I)}\quad \frac{P^{2(\kappa_1-\kappa_2)}\cdot\phi_1(p_2, p_3, p_4)}{\phi_2(p_2, p_3, p_4)} \quad\text{or}\quad \text{(II)}\quad \frac{\phi_1(p_2, p_3, p_4)}{P^{2(\kappa_2-\kappa_1)}\cdot\phi_2(p_2, p_3, p_4)},$$

in which both numerator and denominator are absolute invariants under $G_{24}^{(1)}$. [Art. 16.]

If, then, either numerator or denominator alone is a relative invariant under $G_{24}^{(1)}$, the fraction will have one of the forms

$$\text{(III)}\quad \frac{P^{2\kappa_1+1}\cdot\phi_1(p_2, p_3, p_4)}{\phi_2(p_2, p_3, p_4)},\quad \text{(IV)}\quad \frac{\phi_1(p_2, p_3, p_4)}{P^{2\kappa_2+1}\cdot\phi_2(p_2, p_3, p_4)}.$$

Forms of the types (III) and (IV) will be considered at the end of the paper [Art. 52].

In the succeeding investigation, the forms considered for numerator and denominator of invariant fractions will be absolute invariants under $G_{24}^{(1)}$, so that every such fraction will be of the type

$$\text{(V)}\quad \frac{\phi_1(p_2, p_3, p_4)}{\phi_2(p_2, p_3, p_4)}.$$

20. It follows at once that an invariant fraction cannot have its numerator and denominator of odd degree in z_1, z_2, z_3, for it would then have the form

$$\frac{p_3^{2\kappa_1+1}\cdot\phi_1(p_2, p_3^2, p_4)}{p_3^{2\kappa_2+1}\cdot\phi_2(p_2, p_3^2, p_4)},$$

since p_3 is the only fundamental invariant form of *odd* degree in z_1, z_2, z_3 under the linear subgroup $G_{24}^{(1)}$.

But such a fraction, when reduced, becomes, according as $\kappa_1 \gtrless \kappa_2$,

$$\text{(VI)}\quad \frac{p_3^{2(\kappa_1-\kappa_2)}\cdot\phi_1(p_2, p_3^2, p_4)}{\phi_2(p_2, p_3^2, p_4)} \quad\text{or}\quad \text{(VII)}\quad \frac{\phi_1(p_2, p_3^2, p_4)}{p_3^{2(\kappa_2-\kappa_1)}\cdot\phi_2(p_2, p_3^2, p_4)},$$

in which both terms are of *even* degree in z_1, z_2, z_3.

Thus, *numerator and denominator of an invariant fraction must be rational, integral functions of* p_2, p_3^2, p_4, *and hence also of* p, q, r, *and, therefore, symmetric functions of* z_1, z_2, z_3 owing to the relations (5) and (6), Art. 15.

21. From these considerations come the following definitions:

(a). *An invariant under a quadratic operator of* G_{120} *is a fraction whose numerator and denominator are rational integral functions of* z_1, z_2, z_3 *such that it is transformed into itself by the operator, after throwing off a factor in the z's common to numerator and denominator.*

(b). *An invariant under the Group* G_{120} *is a fraction which is invariant under every one of a system of generators of the group, and so under every transformation of the group.* For this purpose we use the generators of $G_{24}^{(1)}$,

$$K \sim (34), \quad L \sim (23)(45), \quad M \sim (45),$$

and as the extender to G_{120}, the quadratic inversion,

$$T' \sim (12), \qquad \text{[Arts. 4, 28, Part First.]}$$

(c). Any homogeneous function of z_1, z_2, z_3, which is suitable to form the numerator or denominator of an invariant fraction as above defined, is called an *invariant form.* Evidently such a form may be composed of factors or terms,* each of which is a simpler *invariant form.*

(d). Those forms by means of which all other invariant forms of the group can be rationally and integrally expressed, are called the *system of fundamental forms*, or the *complete form-system* of the group.

Theorems on Invariant Forms. Arts. 22–25.

Theorem I.

22. *The most general invariant form under* G_{120} *is of degree* $6n$ *in* z_1, z_2, z_3 *and throws off the factor* r^{2n} *under the quadratic generator* T'.

Proof. We denote by $f_s(z_1, z_2, z_3)$ any homogeneous function of degree s in z_1, z_2, z_3, which is invariant under G_{120} and investigate the value of s and the nature of the factor thrown off when f_s is operated upon by the quadratic extender,

$$T' \sim (12)\,; \quad z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2. \tag{1}$$

* In the case of invariant terms, of course all must have the same "throw-off."

Thus we have* $$f_s(z_1, z_2, z_3)_{T'} = f_{2s}(z_2z_3, z_1z_3, z_1z_2). \tag{2}$$

Since f_s is an invariant form by hypothesis, some factor of degree s [Art. 18] in z_1, z_2, z_3, must be thrown off, thus

$$f_{2s}(z_2z_3, z_1z_3, z_1z_2) = f_s(z_1, z_2, z_3)\,\phi_s(z_1, z_2, z_3). \tag{3}$$

Now apply T' again to (3) and as the left becomes homogeneous of degree $4s$, we have

$$(z_1z_2z_3)^s \cdot f_s(z_1, z_2, z_3) = f_s(z_1, z_2, z_3)\,\phi_s(z_1, z_2, z_3)\,\phi_s(z_1, z_2, z_3)_{T'}. \tag{4}$$

Hence, dividing by $f_s(z_1, z_2, z_3)$,

$$(z_1z_2z_3)^s, = r^s, = \phi_s(z_1, z_2, z_3)\,\phi_s(z_1, z_2, z_3)_{T'}. \tag{5}$$

Therefore, $$\phi_s(z_1, z_2, z_3) = r^t \qquad (t \leqq s) \tag{6}$$

and $$\phi_s(z_1, z_2, z_3)_{T'} = r^{2t}. \tag{7}$$

Substituting (6) and (7) in (5),

$$r^s = r_{3t}, \tag{8}$$

giving $$s = 3t. \tag{9}$$

Then $$s \equiv 0 \pmod 3.$$

But $$s \equiv 0 \pmod 2. \qquad \text{[Art. 20.]}$$

Hence, $$s \equiv 0 \pmod 6. \tag{10}$$

Then from (6),

$$r^t = \phi_s = \phi_{6n} = (z_1z_2z_3)^t,$$

from which it follows, $$t = 2n. \tag{11}$$

Therefore, the general invariant form, R, is of degree $6n$, and throws off the factor r^{2n} under T'.

Theorem II.

23. *The most general invariant form may be decomposed into two factors,*

$$R_{6n} = P^{2\mu} \cdot R_{6(n-2\mu)},$$

where μ *equals zero or a positive integer and* $R_{6(n-2\mu)}$ *contains no factor of* P.

---

* For convenience, the operator is here written as a subscript to the operand.

Proof: The six factors of P are the six sides of the quadrangle Π_1. [(11), Art. 17.] Since these form a conjugate system under $G_{24}^{(1)}$, and since R_{6n} is invariant in the ordinary sense under all transformations of $G_{24}^{(1)}$, it follows that if R_{6n} contains *any* factor of P, it must contain *every* such factor, and if any of these factors are repeated all must be repeated equally often.

Moreover, since we are considering only such forms as are *absolute* invariants under $G_{24}^{(1)}$, such a factor cannot involve P to an *odd* power. [Art. 19.]

Hence, R_{6n} contains as a factor either no factor of P or else $P^{2\mu}$, where μ is a positive integer.

Theorem III.

24. *The curve* $$R_{6n}=P^{2\mu}\cdot R_{6(n-2\mu)}=0, \tag{1}$$

has a multiple point of order $2(n+\mu)$ *at each of the critical points* $1i$, $(i=2 \ldots. 5)$.

Proof: Consider the two factors of (1).

(a). $P^{2\mu}$ fulfills all the conditions for an *invariant form*, namely:

It is an absolute invariant under the linear subgroup $G_{24}^{(1)}$. [Art. 16.]

It is of requisite degree in z_1, z_2, z_3; that is, $6\cdot 2\mu$ according to theorem I.

It throws off the proper factor under T', namely, $r^{2\cdot 2\mu}$, thus

$$P_{T'}^{2\mu}=[z_1^2z_2^2z_3^2(z_1-z_2)^2(z_1-z_3)^2(z_2-z_3)^2]^{\mu}=r^{4\mu}\cdot P^{2\mu}.$$

The curve $$P=0$$

is a degenerate sextic, having three branches through each of the points $1i$, since it represents the six sides of the quadrangle Π_1. [Art. 17.] Hence,

$$P^{2\mu}=0$$

has at each of these points a 6μ-ple point.

(b). Hence, the remaining factor

$$R_{6(n-2\mu)}$$

must be an *invariant form*, since the product R_{6n} is such by hypothesis. That is,

$$[R_{6(n-2\mu)}]_{T'}=r^{2(n-2\mu)}\cdot R_{6(n-2\mu)}. \tag{2}$$

From this it follows that the curve

$$R_{6(n-2\mu)}=0 \tag{3}$$

has the multiple points $1i$ each of order $2(n-2\mu)$.

For if the curve (3) *contains any of the points* $1i$, *it must contain all of them*, since they form a conjugate system under $G_{24}^{(1)}$, under which subgroup, of course, $R_{6(n-2\mu)}$ is invariant.

25. *That it does contain three of these points* follows from the properties of the quadratic transformation T', whose critical points are the three coordinate vertices and whose critical lines form the triangle of reference [Art. 10, Part First], namely:

Under T' a curve through one vertex goes into another curve through the same vertex and throws off as an extra factor the opposite coordinate side; and, conversely, a curve under T' can throw off a coordinate side as a factor only when it contains the opposite coordinate vertex [special case of Art. 11, Part First].

Now (3) reproduces itself under T' and throws off the factor

$$r^{2(n-2\mu)}$$

which contains all three coordinate sides. Hence the curve contains *each of the coordinate vertices and, therefore, all four critical points* as just shown.

Since a curve passing *once* through each of the coordinate vertices throws off, under T', precisely the factor r, in order to throw off $r^{2(n-2\mu)}$, a curve must have $2(n-2\mu)$ branches through each vertex.

Thus *three* (and hence all *four*) of the critical points are multiple points of order $2(n-2\mu)$ on the curve (3).

Therefore, the curve (1)

$$P^{2\mu}\cdot R_{6(n-2\mu)}=0$$

has each of these points as a multiple point of order

$$[6\mu+2(n-2\mu)]=2(n+\mu). \tag{4}$$

§4.—COMPLETE DETERMINATION OF R_{6n} FOR $n=1, 2, 3$.

General Forms of Degree 6, 12, 18. Arts. 26–35.

26. The determination of the most general invariant forms of any given degree $6n$ suitable for numerator and denominator of invariant fractions, involves three steps.

(1). Set up the most general form of the given degree invariant under $G_{24}^{(1)}$, involving arbitrary coefficients.

(2). Apply the transformation T' by which $G_{24}^{(1)}$ extends to G_{120}.

(3). Determine the arbitrary coefficients in such a way that the required factor, r^{2n}, may divide out and leave the original form.

27. For this purpose, all forms will be expressed in terms of p, q, r, since the application of T' to these is peculiarly simple, namely,

$$\left.\begin{aligned} p_{T'} &= q, \\ q_{T'} &= pr, \\ r_{T'} &= r^2. \end{aligned}\right\} \qquad (1)$$

28. The most general invariant form of degree 6 under $G_{24}^{(1)}$ is

$$a_1 p_2^3 + a_2 p_3^2 + a_3 p_2 p_4,$$

which may be expressed in terms of p, q, r in the following manner:*

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	p^6	p^4q	q^3	p^3r	pqr	r^2	p^2q^2
-2^3a_1	3^3	$-2^3\cdot 3^3$	-2^9	0	0	0	$2^6\cdot 3^2$
$+2^6a_2$	1	-2^3	0	2^4	-2^6	2^6	2^4
$+2a_3$	3^2	$-2^3\cdot 3^2$	0	$2^6\cdot 3$	-2^9	0	2^7
Apply T'	q^6	pq^4r	p^3r^3	q^3r^2	pqr^3	r^4	$p^2q^2r^2$
Divide by r^2	..	....	p^3r	q^3	pqr	r^2	p^2q^2

*In each case the form is written in a rectangular array with the original letters at the top, those resulting from the application of T' next to the bottom, those left after dividing all terms possible by r^{2n} in the bottom row, while the arbitrary coefficients occupy the column at the left and the numerical coefficients are written in the body of the array.

The conditions to be met are—

(1). The new form must throw off r^2.

(2). The resulting quotient must be the same as the original form.

In order to meet these conditions,

(a). The coefficients of p^6 and p^4q must vanish.

(b). The coefficients of like terms in the original and resulting forms may be equated [r^2 having been divided out].

These give only *two independent relations*,

$$2^2 \cdot 3^3 a_1 - 2^5 a_2 - 3^2 a_3 = 0,$$
$$2^5 a_1 - 2^3 a_2 - 3a_3 = 0.$$

From which $$a_1 : a_2 : a_3 = 6 : 9 : 40.$$

Hence the most general invariant of the 6th degree is proportional to

$$2 \cdot 3 p_2^3 + 3^2 p_3^2 + 2^3 \cdot 5 p_2 p_4. \tag{2}$$

This is precisely the form A derived from the quintic. [(12), Art. 5.]

Its value in terms of p, q, r may be read from the table

$$\delta A = 2p^2q^2 - 2 \cdot 3\,(p^3r + q^3) + 19pqr - 3^2r^2, \tag{3}$$

thus agreeing with (7), Art. 16.

29. The most general invariant form of degree 12 under $G_{24}^{(1)}$ is

$$b_1 p_2^6 + b_2 p_3^4 + b_3 p_4^3 + b_4 p_2^4 p_4 + b_5 p_3^2 + b_6 p_2^2 p_4^2 + b_7 p_2 p_3^2 p_4, \tag{4}$$

which, in terms of p, q, r in rectangular array, is

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(9)	(9)
	p^{12}	p^8q^2	$p^{10}q$	p^6q^3	p^9r	p^7qr	q^3r^2	p^3r^3	p^2q^5
$+2^6b_1$	3^6	$2^6\cdot 3^5\cdot 5$	$-2^4\cdot 3^6$	$-2^{11}\cdot 3^3\cdot 5$	0	0	0	0	$-2^{16}\cdot 3^2$
$+2^{12}b_2$	1	$2^5\cdot 3$	-2^4	-2^8	2^5	$-2^7\cdot 3$	0	2^{11}	0
$-b_3$	3^3	$2^8\cdot 3^2$	$-2^4\cdot 3^3$	-2^{12}	$2^6\cdot 3^3$	$-2^{11}\cdot 3^2$	0	0	0
-2^4b_4	3^5	$2^7\cdot 3^3\cdot 7$	$-2^4\cdot 3^5$	$-2^{13}\cdot 3^2$	$2^6\cdot 3^4$	$-2^{11}\cdot 3^3$	0	0	-2^{16}
-2^9b_5	3^3	$2^4\cdot 3^2\cdot 19$	$-2^4\cdot 3^3$	$-2^7\cdot 67\cdot$	$2^4\cdot 3^3$	$-2^6\cdot 3^4$	-2^{15}	0	-2^{13}
$+2^2b_6$	3^4	$2^6\cdot 3^2\cdot 13$	$-2^4\cdot 3^4$	$-2^{11}\cdot 3^2$	$2^7\cdot 3^3$	$-2^{12}\cdot 3^2$	0	0	0
$+2^7b_7$	3^2	$2^4\cdot 53$	$-2^4\cdot 3^2$	$-2^7\cdot 17$	$2^4\cdot 3\cdot 7$	$-2^6\cdot 59$	0	$2^{12}\cdot 3$	0
Apply T'	q^{12}	$p^2q^8r^2$	$pq^{10}r$	$p^3q^6r^3$	q^2r^2	pq^7r^3	p^3r^7	q^3r^6	$p^5q^2r^5$
Divide by r^4	....	....	....	....	....	....	p^3r^3	q^3r^2	p^5q^2r

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$p^5q\,r$	p^4qr^2	pq^4r	q^6	p^6r^2	pqr^3	p^4q^4	$p^2q^2r^2$	p^3q^3r	r^4
0	0	0	2^{18}	0	0	$2^{12}\cdot 3^3\cdot 5$	0	0	0
$2^9\cdot 3$	$-2^{10}\cdot 3$	0	0	$2^7\cdot 3$	-2^{13}	2^8	$2^{11}\cdot 3$	-2^{11}	2^{12}
$2^{14}\cdot 3$	$-2^{16}\cdot 3$	0	0	$2^{12}\cdot 3^2$	0	0	0	0	0
$2^{13}\cdot 3^3$	0	2^{18}	0	0	0	$2^{12}\cdot 3^3$	0	$-2^{17}\cdot 3$	0
$2^9\cdot 3^2\cdot 5$	$-2^9\cdot 3^3$	2^{15}	0	$2^6\cdot 3^3$	0	$2^{10}\cdot 13$	$2^{12}\cdot 3^2$	$-2\ \cdot 11$	0
$2^{13}\cdot 3\cdot 5$	$-2^{16}\cdot 3$	0	0	$2^{12}\cdot 3^2$	0	2^{14}	2^{18}	-2^{17}	0
$2^9\cdot 3^3$	$-2^9\cdot 7^2$	0	0	$2^6\cdot 3\cdot 19$	-2^{15}	2^{11}	$2^{13}\cdot 5$	-2^{14}	0
$p^2q^5r^4$	pq^4r^5	p^4qr^6	p^6r^6	q^6r^4	pqr^7	$p^4q^4r^4$	$p^2q^2r^6$	$p^3q^3r^5$	r^8
p^2q^5	pq^4r	p^4qr^2	p^6r^2	q^6	pqr^3	p^4q^4	$p^2q^2r^2$	p^3q^3r	r^4

30. The conditions to be met here are—

(a). The coefficients of terms (1) to (6) must vanish.

(b). The coefficients of term (7), (8); (9), (10); (11), (12); (13), (14), must be equal in pairs.

(c). The coefficients of terms (15 to (19) are the same in the old and new forms, and hence give no relations.

There are then 10 relations, which are not all independent, but readily reduce to the following 7 equations:

b_1	b_2	b_3	b_4	b_5	b_6	b_7		
$2^6 \cdot 3^6$	$+2^{12}$	-3^3	$-2^4 \cdot 3^5$	$-2^9 \cdot 3^3$	$+2^2 \cdot 3^4$	$+2^7 \cdot 3^2$	$=0$	(5)
$2^4 \cdot 3^5$	$+2^9 \cdot 3$	-3^2	$-2^3 \cdot 3^3 \cdot 7$	$-2^5 \cdot 3^2 \cdot 19$	$+3^2 \cdot 13$	$+2^3 \cdot 53$	$=0$	
$2^5 \cdot 3^5 \cdot 5$	$+2^8$	-1	$-2^5 \cdot 3^2$	$-2^4 \cdot 67$	$+2 \cdot 3^2$	$+2^2 \cdot 17$	$=0$	
2^{12}	$-2^7 \cdot 3$	$+3^2$	0	$+2^3 \cdot 3^3$	$-2^2 \cdot 3^2$	$-2 \cdot 3 \cdot 19$	$=0$	
0	$+2^5$	-1	0	-2^6	0	$+2 \cdot 3$	$=0$	
$2^8 \cdot 3^2$	$+2^7 \cdot 3$	-3	$-2^3 \cdot 5 \cdot 7$	$-2^4 \cdot 61$	$+2 \cdot 3 \cdot 5$	$+2^2 \cdot 3^3$	$=0$	
0	$-2^6 \cdot 3$	$+3$	$+2^6$	$+2^2 \cdot 7 \cdot 13$	$-2^2 \cdot 3$	-7^2	$=0$	

31. Two invariants of the 12th degree are already known, P^2 [Art. 24], and A^2 [Art. 28].

Hence their values,

	p_2^6	p_3^4	p_4^3	$p_2^4 p_4$	$p_2^2 p_3^2$	$p_2^2 p_2^4$	$p_2 p_3^2 p_4$
$A^2 =$	$2^2 \cdot 3^2$	$+ 3^4$	0	$+ 2^5 \cdot 3 \cdot 5$	$+ 2^2 \cdot 3^3$	$+ 2^6 \cdot 5^2$	$+ 2^4 \cdot 3^2 \cdot 5$
$P^2 =$	0	$- 3^3$	$+ 2^8$	$+ 2^4$	$- 2^2$	$- 2^7$	$+ 2^4 \cdot 3^2$
	b_1	b_2	b_3	b_4	b_5	b_6	b_7

(6)

must each satisfy the above system of equations. This is easily verified. Therefore, there exists an infinity of solutions of the system (5) of the form

$$m_1 A^2 + m_2 P^2, \tag{7}$$

where m_1 and m_2 are arbitrary parameters. Hence, all the first minors in the determinant of (5) must vanish, as well as the determinant itself. If, now, any second minor does not vanish, then no solution exists other than those included in the form (7). The second minor obtained by omitting the 5th and 7th columns and the 1st and 2d rows is easily shown to be different from zero.

Hence, all invariant forms of the 12*th degree are included in the form* (7).

32. The most general form of degree 18 invariant under $G_{24}^{(1)}$ involves 12 arbiträry constants, thus

$$c_1 p_2^9 + c_2 p_3^6 + c_3 p_2^6 p_3^2 + c_4 p_3^2 p_4^3 + c_5 p_2^3 p_3^4 + c_6 p_2^7 p_4 + c_7 p_2^5 p_4^2 + c_8 p_2^3 p_4^3$$
$$+ c_9 p_2 p_4^4 + c_{10} p_2 p_3^4 p_4 + c_{11} p_2^4 p_3^2 p_4 + c_{12} p_2^2 p_3^2 p_4^2. \tag{8}$$

When expressed in terms of p, q, r in the rectangular array, this becomes

	(1)	(2)	(3)	(4)	(5)	(6)
	p^{18}	$p^{16}q$	$p^{15}r$	$p^{14}q^2$	$p^{12}q^3$	$p^{12}r^2$
-2^9c_1	3^9	$-2^3\cdot 3^{10}$	0	$2^8\cdot 3^9$	$-2^{11}\cdot 3^7\cdot 7$	0
$+2^{18}c_2$	1	$-2^3\cdot 3$	$2^4\cdot 3$	$2^4\cdot 3\cdot 5$	$-2^8\cdot 5$	$2^6\cdot 3\cdot 5$
$+2^{12}c_3$	3^6	$-2^3\cdot 3^7$	$2^4\cdot 3^6$	$2^4\cdot 3^5\cdot 47$	$-2^9 . 3^5\cdot 5$	$2^6\cdot 3^6$
-2^6c_4	3^3	$-2^3\cdot 3^4$	$2^4\cdot 3^3\cdot 5$	$2^4\cdot 3^2\cdot 43$	$-2^8\cdot 5\cdot 23$	$2^6\cdot 3^2\cdot 5\cdot 23$
$-2^{15}c_5$	3^3	$-2^3\cdot 3^4$	$2^5\cdot 3^3$	$2^5\cdot 3^2\cdot 23$	$-2^9\cdot 73$	$2^7\cdot 3^4$
$+2^7c_6$	3^8	$-2^3\cdot 3^9$	$2^6\cdot 3^7$	$2^6\cdot 3^6\cdot 5\cdot 7$	$-2^9\cdot 3^5\cdot 7\cdot 11$	0
-2^5c_7	3^7	$-2^3\cdot 3^8$	$2^7\cdot 3^6$	$2^7\cdot 3^5\cdot 17$	$-2^{10}\cdot 3^4\cdot 5\cdot 7$	$2^{12}\cdot 3^5$
$+2^3c_8$	3^6	$-2^3\cdot 3^7$	$2^6\cdot 3^6$	$2^6\cdot 3^5\cdot 11$	$-2^9\cdot 3^5\cdot 7$	$2^{12}\cdot 3^5$
$-2c_9$	3^5	$-2^3\cdot 3^6$	$2^8\cdot 3^4$	$2^{11}\cdot 3^3$	$-2^{12}\cdot 3^2\cdot 7$	$2^{13}\cdot 3^4$
$-2^{13}c_{10}$	3^2	$-2^3\cdot 3^3$	$2^5\cdot 3\cdot 5$	$2^5\cdot 67$	$-2^{10}\cdot 11$	$2^7\cdot 3\cdot 5^2$
$-2^{10}c_{11}$	3^5	$-2^3\cdot 3^6$	$2^4\cdot 3^4\cdot 7$	$2^4\cdot 3^3\cdot 137$	$-2^8\cdot 3^2\cdot 143$	$2^6\cdot 3^{14}\cdot 19$
$+2^8c_{12}$	3^4	$-2^3\cdot 3^5$	$2^4\cdot 3^3\cdot 11$	$2^4\cdot 3^2\cdot 7\cdot 19$	$-2^8\cdot 3^2\cdot 43$	$2^6\cdot 3^3\cdot \overline{13}^2$
Apply T'	q^{18}	$pq^{16}r$	$q^{15}r^2$	$p^2q^{14}r^2$	$p^3q^{12}r^3$	$q^{12}r^4$
Divide by r^6	....	......	......	....	....	....

	(7)	(8)	(9)	(10)	(11)
	$p^{10}q^4$	p^8q^5	$p^{13}qr$	$p^{11}q^2r$	$p^{10}qr^2$
-2^9c_1	$2^{13}\cdot 3^7\cdot 7$	$-2^{16}\cdot 3^6\cdot 7$	0	0	0
$+2^{18}c_2$	$2^8\cdot 3\cdot 5$	$-2^{11}\cdot 3$	$-2^6\cdot 3\cdot 5$	$2^9\cdot 3\cdot 5$	$-2^{10}\cdot 3^5$
$+2^{12}c_3$	$2^{10}\cdot 3^3\cdot 5\cdot 29$	$-2^{20}\cdot 3^2$	$-2^6\cdot 3^6\cdot 5$	$2^{13}\cdot 3^5$	$-2^{10}\cdot 3^6$
-2^6c_4	$2^{12}\cdot 17$	-2^{16}	$-2^6\cdot 3^2\cdot 71$	$2^{11}\cdot 3\cdot 47$	$-2^{10}\cdot 3\cdot 7\cdot 43$
$-2^{15}c_5$	$2^8\cdot 491$	$-2^{11}\cdot 3\cdot 41$	$-2^7\cdot 3^3\cdot 5$	$2^9\cdot 3^2\cdot 31$	$-2^{11}\cdot 3^4$
$+2^7c_6$	$2^{12}\cdot 3^5\cdot 5\cdot 7$	$-2^{15}\cdot 3^3\cdot 7\cdot 13$	$-2^9\cdot 3^6\cdot 7$	$2^{12}\cdot 3^6\cdot 7$	0
-2^5c_7	$2^{12}\cdot 3^3\cdot 5\cdot 17$	$-2^{15}\cdot 3^2\cdot 61$	$-2^{10}\cdot 3^5\cdot 7$	$2^{15}\cdot 3^4\cdot 5$	$-2^{15}\cdot 3^4\cdot 5$
$+2^3c_8$	$2^{13}\cdot 3^3\cdot 11$	$-2^{17}\cdot 3^3$	$-2^9\cdot 3^5\cdot 7$	$2^{12}\cdot 3^4\cdot 19$	$-2^{15}\cdot 3^4\cdot 5$
$-2c_9$	$2^{16}\cdot 3^2$	-2^{19}	$-2^{11}\cdot 3^3\cdot 7$	$2^{15}\cdot 3^4$	$-2^{16}\cdot 3^3\cdot 5$
$-2^{13}c_{10}$	$2^8\cdot 3\cdot 43$	$-2^{11}\cdot 5^2$	$-2^7\cdot 73$	$2^9\cdot 3\cdot 47$	$-2^{11}\cdot 71$
$-2^{10}c_{11}$	$2^{11}\cdot 3^2\cdot 59$	$-2^{15}\cdot 5\cdot 13$	$-2^6\cdot 3^3\cdot 7\cdot 11$	$2^{11}\cdot 3^3\cdot 5^2$	$-2^{10}\cdot 3^3\cdot 53$
$+2^8c_{12}$	$2^{10}\cdot 277$	$-2^{15}\cdot 13$	$-2^6\cdot 3^2\cdot 157$	$2^{12}\cdot 3\cdot 5\cdot 11$	$-2^{10}\cdot 3\cdot 11\cdot 41$
Apply T'	$p^4q^{10}r^4$	$p^5q^8r^5$	$pq^{13}r^3$	$p^2q^{11}r^4$	$pq^{10}r^5$
Divide by r^6	....	....	....	....	....

	(12)	(13)	(14)	(15)	(16)	(17)	(18)
	p^9q^3r	q^9	p^9r^3	p^6r^4	q^6r^2	p^4q^7	p^7q^4r
$-2^{9}c_1$	0	-2^{27}	0	0	0	$-2^{23}\cdot 3^4$	0
$+2^{18}c_2$	$-2^{11}\cdot 3\cdot 5$	0	$2^{11}\cdot 5$	$2^{12}\cdot 3\cdot 5$	0	0	$2^{12}\cdot 3\cdot 5$
$+2^{12}c_3$	$-2^{12}\cdot 3\cdot 5\cdot 17$	0	0	0	2^{24}	$-2^{20}\cdot 11\cdot$	$2^{16}\cdot 3^4\cdot 5$
$-2^{6}c_4$	$-2^{14}\cdot 5\cdot 11$	0	$2^{12}\cdot 5\cdot 47$	$2^{18}\cdot 5^2$	0	0	2^{20}
$-2^{15}c_5$	$-2^{11}\cdot 5\cdot 61$	0	$2^{11}\cdot 3^3$	$2^{12}\cdot 3^3$	0	-2^{17}	$2^{14}\cdot 3\cdot 31$
$+2^{7}c_6$	$-2^{15}\cdot 3^4\cdot 5\cdot 7$	0	0	0	0	$-2^{21}\cdot 3^2\cdot 5$	$2^{18}\cdot 3\cdot 5\cdot 7$
$-2^{5}c_7$	$-2^{17}\cdot 3^4\cdot 5$	0	0	0	0	-2^{23}	$2^{19}\cdot 3^2\cdot 5$
$+2^{3}c_8$	$-2^{15}\cdot 3^3\cdot 5^2$	0	$2^{18}\cdot 3^3$	0	0	0	$2^{22}\cdot 3^2$
$-2c_9$	$-2^{19}\cdot 3\cdot 5$	0	$2^{20}\cdot 3^2$	$2^{24}\cdot 3$	0	0	2^{23}
$-2^{13}c_{10}$	$-2^{11}\cdot 3^3\cdot 5$	0	$2^{11}\cdot 3^2\cdot 5$	$2^{12}\cdot 3\cdot 5\cdot 7$	0	0	2^{19}
$-2^{10}c_{11}$	$-2^{13}\cdot 3\cdot 5\cdot 47$	0	$2^{12}\cdot 3^4$	0	0	-2^{20}	$2^{16}\cdot 5\cdot 41$
$+2^{8}c_{12}$	$-2^{12}\cdot 5\cdot \overline{11}^2$	0	$2^{13}\cdot 3^2\cdot 11$	$2^{18}\cdot 3^2$	0	0	$2^{18}\cdot 17$
Apply T'	$p^3q^9r^5$	p^9r^9	q^9r^6	q^6r^8	p^6r^{10}	$p^7q^4r^7$	$p^4q^7r^6$
Divide by r^6	....	p^9r^3	q^9	q^6r^2	p^6r^4	p^7q^4r	p^4q^7

	(19)	(20)	(21)	(22)	(23)	(24)
	p^3r^5	q^3r^4	p^2q^8	$p^8q^2r^2$	p^7qr^3	pq^7r
$-\,2^9c_1$	0	0	$2^{24}\cdot 3^3$	0	0	0
$+\,2^{18}c_2$	$2^{16}\cdot 3$	0	0	$2^{11}\cdot 3^2\cdot 5$	$-\,2^{12}\cdot 3\cdot 5$	0
$+\,2^{12}c_3$	0	0	2^{22}	$2^{12}\cdot 3^5\cdot 5$	0	$-\,2^{24}$
$-\,2^6c_4$	2^{24}	0	0	$2^{14}\cdot 3^2\cdot 29$	$-\,2^{17}\cdot 67$	0
$-\,2^{15}c_5$	0	$-\,2^{21}$	0	$2^{11}\cdot 3^3\cdot 19$	$-\,2^{13}\cdot 3^4$	0
$+\,2^7c_6$	0	0	2^{26}	0	0	$-\,2^{27}$
$-\,2^5c_7$	0	0	0	$2^{19}\cdot 3^3\cdot 5$	0	0
$+\,2^3c_8$	0	0	0	$2^{18}\cdot 3^5$	$-\,2^{21}\cdot 3^3$	0
$-\,2c_9$	0	0	0	$2^{22}\cdot 3^2$	$-\,2^{23}\cdot 3^2$	0
$-\,2^{13}c_{10}$	$2^{18}\cdot 3$	0	0	$2^{11}\cdot 3\cdot 7\cdot 19$	$-\,2^{13}\cdot 3\cdot 41$	0
$-\,2^{10}c_{11}$	0	0	0	$2^{13}\cdot 3^4\cdot 13$	$-\,2^{17}\cdot 3^3$	0
$+\,2^8c_{12}$	0	0	0	$2^{12}\cdot 5\cdot 353$	$-\,2^{19}\cdot 3\cdot 5$	0
Apply T'	q^3r^{10}	p^3r^7	$p^8q^2r^8$	$p^2q^8r^6$	pq^7r^7	p^7qr^9
Divide by r^6	q^3r^4	p^3r	$p^8q^2r^2$	p^2q^8	pq^7r	p^7qr^3

	(25)	(26)	(27)	(28)	(29)	(30)	(31)
	$p^6q^3r^2$	p^3q^6r	$p^5q^2r^3$	$p^2q^5r^2$	p^4qr^4	pq^4r^3	r^6
-2^9c_1	0	0	0	0	0	0	0
$+2^{18}c_2$	$-2^{14}\cdot 3\cdot 5$	0	$2^{15}\cdot 3\cdot 5$	0	$-2^{15}\cdot 3\cdot 5$	0	2^{18}
$+2^{12}c_3$	$-2^{17}\cdot 3^3\cdot 5$	$2^{23}\cdot 5$	0	$-2^{22}\cdot 3^2$	0	0	0
-2^6c_4	$-2^{18}\cdot 5^2$	0	$2^{20}\cdot 19$	0	$-2^{22}\cdot 7$	0	0
$-2^{15}c_5$	$-2^{14}\cdot 3\cdot 67$	2^{20}	$2^{16}\cdot 3^2\cdot 5$	$-2^{20}\cdot 3$	$-2^{15}\cdot 3^3$	2^{22}	0
$+2^7c^6$	0	$2^{24}\cdot 3\cdot 7$	0	0	0	0	0
-2^5c_7	$-2^{22}\cdot 3^2\cdot 5$	2^{26}	0	-2^{27}	0	0	0
$+2^3c_8$	$-2^{21}\cdot 3^2\cdot 7$	0	$2^{24}\cdot 3^2$	0	0	0	0
$-2c_9$	$-2^{24}\cdot 3$	0	2^{27}	0	-2^{27}	0	0
$-2^{13}c_{10}$	$-2^{14}\cdot 3\cdot 41$	0	$2^{16}\cdot 5\cdot 11$	0	$-2^{15}\cdot 89$	0	0
$-2^{10}c_{11}$	$-2^{23}\cdot 3$	2^{23}	$2^{11}\cdot 3^3$	$-2^{22}\cdot 5$	0	2^{24}	0
$+2^{18}c_{12}$	$-2^{17}\cdot 5^3$	0	$2^{19}\cdot 47$	0	$-2^{22}\cdot 3$	0	0
Apply T'	$p^3q^6r^7$	$p^6q^3r^8$	$p^2q^5r^8$	$p^5q^2r^9$	pq^4r^9	p^4qr^{10}	r^{12}
Divide by r^6	p^3q^6r	$p^6q^3r^2$	$p^2q^5r^2$	$p^5q^2r^3$	pq^4r^3	p^4qr^4	r^6

	(32)	(33)	(34)	(35)	(36)	(37)
	p^6q^6	p^5q^5r	$p^4q^4r^2$	$p^3q^3r^3$	$p^2q^2r^4$	pqr^5
-2^9c_1	$2^{20}\cdot3^4\cdot7$	0	0	0	0	0
$+2^{18}c_2$	2^{12}	$-2^{14}\cdot3$	$2^{14}\cdot3\cdot5$	$-2^{17}\cdot5$	$2^{16}\cdot3\cdot5$	$-2^{18}\cdot3$
$+2^{12}c_3$	$2^{16}\cdot211$	$-2^{18}\cdot3^2\cdot19$	$2^{18}\cdot3\cdot5$	0	0	0
-2^6c_4	0	0	0	0	0	0
$-2^{15}c_5$	$2^{14}\cdot17$	$-2^{17}\cdot3\cdot5$	$2^{17}\cdot3\cdot13$	$-2^{19}\cdot11$	$2^{18}\cdot3^2$	0
$+2^7c_6$	$2^{18}\cdot3^2\cdot7^2$	$-2^{21}\cdot3^3\cdot7$	0	0	0	0
-2^5c_7	$2^{21}\cdot3^2$	$-2^{22}\cdot3\cdot11$	$2^{24}.3.5$	0	0	0
$+2^3c_8$	2^{21}	$-2^{23}\cdot3$	$2^{25}\cdot3$	-2^{27}	0	0
$-2c_9$	0	0	0	0	0	0
$-2^{13}c_{10}$	2^{15}	$-2^{17}\cdot3$	$2^{18}\cdot7$	-2^{22}	$2^{19}\cdot3^2$	-2^{21}
$-2^{10}c_{11}$	$2^{16}\cdot5\cdot7$	$-2^{18}\cdot3^2\cdot7$	$2^{18}\cdot139$	$-2^{23}\cdot3$	0	0
$+2^8c_{12}$	2^{18}	$-2^{20}\cdot3$	$2^{20}\cdot13$	$-2^{23}\cdot3$	2^{24}	0
Apply T'	$p^6q^6r^6$	$p^5q^5r^7$	$p^4q^4r^8$	$p^3q^3r^9$	$p^2q^2r^{10}$	pqr^{11}
Divide by r^6	p^6q^6	p^5q^5r	$p^4q^4r^2$	$p^3q^3r^3$	$p^2q^2r^4$	pqr^5

33. The conditions in this case are—

(a). The coefficients of terms (1) to (12) must vanish.

(b). The coefficients of terms (13) to (30) must be equal in pairs; that is,

(13), (14); (15), (16); .... (29), (30).

(c). The coefficients of terms (31) to (37) are the same in the new form as in the original, and hence afford no relations.

These conditions give rise to 21 relations among the 12 c's. This system reduces readily to the following 16 equations:

	c_1	c_2	c_3	c_4	c_5	c_6
(1)	$2^8 \cdot 3^9$	-2^{17}	$-2^{11} \cdot 3^6$	$+2^5 \cdot 3^3$	$+2^{14} \cdot 3^3$	$-2^6 \cdot 3^8$
(2)	0	$+2^{13}$	$+2^7 \cdot 3^5$	$-2 \cdot 3^2 \cdot 5$	$-2^{11} \cdot 3^2$	$+2^4 \cdot 3^6$
(3)	$2^8 \cdot 3^7 \cdot 7$	$-2^{14} \cdot 5$	$-2^8 \cdot 3^3 \cdot 157$	$+2^2 \cdot 5 \cdot 23$	$+2^{12} \cdot 73$	$-2^4 \cdot 3^5 \cdot 7 \cdot 11$
(4)	0	$+2^{12} \cdot 5$	$+2^6 \cdot 3^5$	$-3 \cdot 5 \cdot 23$	$-2^{10} \cdot 3^3$	0
(5)	$2^7 \cdot 3^6 \cdot 7$	$-2^{11} \cdot 3$	$-2^{14} \cdot 3^2$	$+2^4$	$+2^8 \cdot 3 \cdot 41$	$-2^4 \cdot 3^3 \cdot 7 \cdot 13$
(6)	$-2^6 \cdot 3^7 \cdot 7$	$+2^{10} \cdot 3 \cdot 5$	$+2^6 \cdot 3^3 \cdot 5 \cdot 29$	$-2^2 \cdot 17 \cdot$	$-2^7 \cdot 491$	$+2^3 \cdot 3^5 \cdot 5 \cdot 7$
(7)	0	$-2^{12} \cdot 3 \cdot 5$	$-2^6 \cdot 3^6$	$+3 \cdot 7 \cdot 43$	$+2^{10} \cdot 3^4$	0
(8)	-2^{18}	$+2^{11} \cdot 5$	0	$-5 \cdot 47$	$-2^8 \cdot 3^3$	0
(9)	0	$+2^6 \cdot 3 \cdot 5$	-2^{12}	-5^2	$-2^3 \cdot 3^3$	0
(10)	$-2^8 \cdot 3^4$	$+2^6 \cdot 3 \cdot 5$	$+2^4 \cdot 7 \cdot 83$	-2^2	$-2^5 \cdot 101$	$+2 \cdot 3^2 \cdot 5 \cdot 29$
(11)	0	$+2^4 \cdot 3$	0	-1	-2^6	0
(12)	$2^{13} \cdot 3^3$	$+2^9 \cdot 3^2 \cdot 5$	$+2^4 \cdot 191$	$-3^2 \cdot 29$	$-2^6 \cdot 3^3 \cdot 19$	-2^{12}
(13)	0	$-2^8 \cdot 3 \cdot 5$	$+2^{13}$	$+67$	$+2^5 \cdot 3^4$	$+2^{11}$
(14)	0	$-2^8 \cdot 3 \cdot 5$	$-2^5 \cdot 5 \cdot 7 \cdot 13$	$+5^2$	$+2^5 \cdot 5 \cdot 53$	$-2^7 \cdot 3 \cdot 7$
(15)	0	$-2^7 \cdot 3 \cdot 5$	$+2^8 \cdot 3^2$	-19	$-2^5 \cdot 3 \cdot 31$	0
(16)	0	$-2^5 \cdot 3 \cdot 5$	0	$+7$	$+2^2 \cdot 5 \cdot 31$	0

	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	
(1)	$+2^4\cdot 3^7$	$-2^2\cdot 3^6$	$+3^5$	$-2^{12}\cdot 3^2$	$+2^9\cdot 3^5$	$-2^7\cdot 3^4$	$=0$
(2)	$-2^3\cdot 3^8$	$+3^5$	-3^3	$+2^9\cdot 5$	$-2^5\cdot 3^3\cdot 7$	$+2^3\cdot 3^2\cdot 11$	$=0$
(3)	$+2^3\cdot 3^4\cdot 5\cdot 7$	$-3^5\cdot 7$	$+2\cdot 3^2\cdot 7$	$-2^{11}\cdot 11$	$+2^6\cdot 3^2\cdot 143$	$-2^4\cdot 3^2\cdot 43$	$=0$
(4)	$-2^5\cdot 3^4$	$+2^3\cdot 3^4$	$-2^2\cdot 3^3$	$+2^8\cdot 5^2$	$-2^4\cdot 3^3\cdot 19$	$+2^2 . 3\cdot \overline{13}^2$	$=0$
5)	$+2^2\cdot 3^2\cdot 61$	$-2^2\cdot 3^3$	$+2^2$	$-2^6\cdot 5^2$	$+2^7\cdot 5\cdot 13$	$-2^5\cdot 13$	$=0$
(6)	$-2\cdot 3^3\cdot 5\cdot 17$	$+3^3\cdot 11$	$-2\cdot 3^2$	$+2^5\cdot 3\cdot 43$	$-2^5\cdot 3^2\cdot 59$	$2^2\cdot 277$	$=0$
(7)	$+2^4\cdot 3^4 . 5$	$-2^2\cdot 3^4\cdot 5$	$+2\cdot 3^3\cdot 5$	$-2^8\cdot 71$	$+2^4\cdot 3^3\cdot 53$	$-2^2\cdot 3\cdot 11\cdot 41$	$=0$
(8)	0	$+2^3\cdot 3^3$	$-2^3\cdot 3^2$	$+2^6\cdot 3^2\cdot 5$	$-2^4\cdot 3^4$	$+2^3\cdot 3^2\cdot 11$	$=0$
(9)	0	0	$-2\cdot 3$	$+2\cdot 3\cdot 5\cdot 7$	0	$+2^2\cdot 3^2$	$=0$
(10)	-241	$+2\cdot 3^2$	-1	$+2^8$	$-2^2\cdot 13\cdot 17$	$+2^2\cdot 17$	$=0$
(11)	0	0	0	$+2\cdot 3$	0	0	$=0$
(12)	$-2^4\cdot 3^3\cdot 5$	$+2^2\cdot 3^5$	$-2^3\cdot 3^2$	$+2^4\cdot 3\cdot 7\cdot 19$	$-2^3\cdot 3^4\cdot 13$	$+5\cdot 353$	$=0$
(13)	0	$-2\cdot 3^3$	$+2\cdot 3^2$	$-2^3\cdot 3\cdot 41$	$+2^4\cdot 3^3$	$-2^4\cdot 3\cdot 5$	$=0$
(14)	$+2^3\cdot 61$	$-3^2\cdot 7$	$+2\cdot 3$	$-2^3\cdot 3\cdot 41$	$+2^{11}$	$-2\cdot 5^3$	$=0$
(15)	-2^6	$+2\cdot 3^2$	-2^2	$+2^3\cdot 5\cdot 11$	$-2^3\cdot 67$	$+2\cdot 47$	$=0$
(16)	0	0	$+1$	-89	$+2^6$	$-2^2\cdot 3$	$=0$

34. Two invariant forms of degree 18 are already known, A^3 and AP^2, and since these depend upon two of the forms derived in (7), (8), (9), Art. 16, it is at once suggested that the third form C, which is of degree 18, may also belong to this system. [See (15), Art. 12.]

In fact, it is easily verified that the 16 equations are satisfied by these three forms:

	c_1	c_2	c_3	c_4	c_5	c_6
$A^3=$	$+2^3\cdot 3^3$	$+3^6$	$+2^2\cdot 3^5$	0	$+2\cdot 3^6$	$+2^5\cdot 3^3\cdot 5$
$AP^2=$	0	-3^5	$-2^3\cdot 3$	$+2^8\cdot 3^2$	$-2\cdot 3^2\cdot 11$	$+2^5\cdot 3$
$C=$	-2^2	$-2\cdot 3^3\cdot 23$	$-2\cdot 17$	$-2^7\cdot 151$	$-2^2\cdot 73$	-2^6
	p_2^9	p_3^6	$p_2^6 p_3^2$	$p_3^2 p_4^3$	$p_2^3 p_3^4$	$p_2^7 p_4$

c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	
$+2^7.3^2\cdot 5^2$	$+2^9\cdot 5^3$	0	$+2^3\cdot 3^5\cdot 5$	$+2^5\cdot 3^4\cdot 5$	$+2^6\cdot 3^3\cdot 5^2$	
-2^7	$-2^9.7$	$+2^{11}\cdot 5$	$+2^3\cdot 3^3$	$+2^4\cdot 53\cdot$	$+2^9\cdot 3^2$	(9)
$+2^7$	$-2^{10}\cdot 13$	$+2^{10}\cdot 5\cdot 11$	$+2^4\cdot 3^2\cdot 5^2$	$+2^3\cdot 3^3$	$+2^5\cdot 7\cdot 11$	
$p_2^5 p_4^2$	$p_2^3 p_4^3$	$p_2 p_4^4$	$p_2 p_3^4 p_4$	$p_2^4 p_3^2 p_4$	$p_2^2 p_3^2 p_4^2$	

35. It follows that an infinity of solutions of the above system of equations exists of the form

$$m_1 A^3 + m_2 AP^2 + m_3 C, \tag{10}$$

where the m's are arbitrary parameters.

Hence, all determinants of the 12th order in the matrix of the coefficients must vanish and also *all* the first and second minors of these. A third minor, however, is easily found which does not vanish, namely, by depleting the first 7 rows and the 2d, 5th and 10th columns. Thus, no solution exists other than those included in (10).

Therefore, *the only invariant forms of degree* 18 *are included in the general form*

$$m_1 A^3 + m_2 A P^2 + m_3 C.$$

§5.—The System of Fundamental Invariants.

Invariants of the Linear Groups G_6^{1i}. Arts. 36–38.

36. It has been shown that each of the pencils $1i$ ($i = 2 \ldots\ldots 5$) is invariant under a dihedron group G_6^{1i}. [Art. 13.]

The known* complete form-systems of these subgroups will now furnish the means of determining the system of fundamental invariants for G_{120}.

These auxiliary systems may be read by the group properties already shown in the configuration Π, Fig. X, Part First.

It will be sufficient to deduce the system for one pencil, say at the vertex 13.

The three conics $\qquad 13 \cdot 24, \; 13 \cdot 25, \; 13 \cdot 45$

are permuted under the dihedron

$$G_6^{13} \sim \{245\} \text{ all.}$$

Hence, this group must also permute among themselves those directions in the pencil 13 which are given by the corresponding tangents to these conics at that point,

$$\left.\begin{aligned} z_1 - 2z_2 &= 0, \\ 2z_1 - \;\, z_2 &= 0, \\ z_1 + \;\, z_2 &= 0. \end{aligned}\right\} \tag{1}$$

The product of these tangential quantics is, therefore, one of the cubic invariant forms under G_6^{13},

$$f_1 = -2z_1^3 + 3(z_1^2 z_2 + z_1 z_2^2) - 2z_2^3. \tag{2}$$

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* Klein, "Ikosaeder," Kapitel I, §9.

The other cubic invariant is the product of the three sides of the quadrangle Π_1 which pass through the point 13,

$$f_2 = z_1^2 z_2 - z_1 z_2^2. \tag{3}$$

37. In order to find the quadratic invariant under G_6^{13}, we take the invariant conic belonging to the fundamental system of $G_{24}^{(1)}$ [Art. 14],

$$p_2 \equiv -3\Sigma z_1^2 + 2\Sigma z_1 z_2 = 0 \tag{4}$$

and find the tangents to it from the point 13, namely [Art. 46, Part First],

$$\left.\begin{aligned} \omega z_1 + z_2 &= 0, \\ \omega^2 z_1 + z_2 &= 0. \end{aligned}\right\} \tag{5}$$

The product of these tangential quantics is, then, the quadratic invariant form of G_6^{13},

$$f_3 = z_1^2 - z_1 z_2 + z_2^2. \tag{6}$$

This follows since, under G_6^{13}, *the point* 13 *is fixed and the conic is fixed*, so that the tangents must be either fixed or permuted, making their product an invariant. Since there is *only one quadratic invariant* under G_6^{13}, it must be f_3 thus determined.

In like manner the systems may be found for the other pencils and their groups.

38. It will be seen that the above forms f_1, f_2, f_3 correspond to Klein's forms F_1, F_2, F_3 in the following manner:

$$\left.\begin{matrix} F_1 & F_2^2 & F_3^3 \\ f_1 & -3^3 f_2^2 & 2^2 f_3^3 \end{matrix}\right\} \tag{7}$$

so that the identity* holds

$$F_1^2 - F_2^2 - F_3^3 \equiv f_1^2 + 3^3 f_2^2 - 2^2 f_3^3 = 0. \tag{8}$$

For the point 13 we have

$$\left.\begin{aligned} f_1^2 &= 4(z_1^6 + z_2^6) - 12(z_1^5 z_2 + z_1 z_2^5) - 3(z_1^4 z_2^2 + z_1^2 z_2^4) + 26 z_1^3 z_2^3, \\ f_2^2 &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad + (z_1^4 z_2^2 + z_1^2 z_1^4) - 2 z_1^3 z_2^3, \\ f_3^3 &= (z_1^6 + z_2^6) - 3(z_1^5 z_2 + z_1 z_2^5) + 6(z_1^4 z_2^2 + z_1^2 z_2^4) - 7 z_1^3 z_2^3. \end{aligned}\right\} \tag{9}$$

From (9) we see at once that (8) holds.

* Klein, "Ikosaeder," p. 49.

The only remaining type-form of the 6th degree under G_6^{13} is

$$f_1 f_2 = -2(z_1^5 z_2 - z_1 z_2^5) + 5(z_1^4 z_2^2 - z_1^2 z_2^4),^* \tag{10}$$

which alone is non-symmetric in z_1, z_2.

Relation of A, P^2, C to f_1, f_2, f_3. Arts. 39–47.

39. It has been seen by theorems I, II [Arts. 23, 24] that every invariant form under G_{120} may be reduced to

$$R_{6n} = P^{2\mu} \cdot R_{6(n-2\mu)},$$

and that it has at each of the critical points a multiple point of order $2(n+\mu)$.

For the forms already considered the parameters, n and μ have the following special values:

	n	μ	$2(n+\mu)$
A	1	0	2
A^2	2	0	4
A^3	3	0	6
P^2	2	1	6
AP^2	3	2	10
C	3	0	6

(1)

40. Consider the sextic curve

$$A = 0. \tag{2}$$

* See Art. 46.

The product of its tangential quantics at 13, one of its double points, is the binary quadratic invariant for that point. For

$$A = 0$$

is fixed, and the point 13 is fixed under the linear group G_6^{13}. Therefore, the two tangents are either fixed or permuted, and their product is the quadratic invariant f_3, since only one such form exists under G_6^{13}.

In this sense the ternary form A is said to correspond to the binary form f_3, and since A is the only ternary invariant form of the sixth degree, the correspondence is one to one.

If, however, the second polar of (2) with respect to the vertex 13 be formed, f_3 is found to carry a numerical factor, thus

$$A \sim 2f_3. \tag{3}$$

41. The correspondence is very different in the case of the degenerate curve.

$$P^2 = 0, \tag{4}$$

which represents the six sides, each taken twice. The product of these at 13 is exactly f_2^2, so that

$$P^2 \sim f_2^2. \tag{5}$$

Whereas, the other 12th degree form A^2 gives the correspondence

$$A^2 \sim 2^2 f_3^2. \tag{6}$$

This illustrates theorem III [Art. 24], *showing how the factor*

$$P^{2\mu} \textit{ in the form } R_{6n}, \qquad (\mu \neq 0)$$

raises the order of the multiple point by 2μ on account of the degeneracy of the curve (4).

For this reason the correspondence (5) is *not one to one*, since f_2^2 may go equally well with some form of degree 18. See (10), Art. 43.

42. In order to find the binary correspondent to the curve

$$C = 0, \tag{7}$$

we form the 6th polar of (7) with reference to the point

$$13;\ 0:0:1. \tag{8}$$

For this purpose the following remarks are useful:

(a).
$$\frac{\partial p}{\partial z_i} = 1, \quad \frac{\partial q}{\partial z_i} = z_j + z_k, \quad \frac{\partial r}{\partial z_i} = z_j z_k,$$
$$(i, j, k = 1, 2, 3).$$

(b). At the point (8),
$$p = 1, \quad q = 0, \quad r = 0.$$

(c). Neither the factor p^α nor any of its derivatives below the $(\alpha + 1)^{\mathrm{st}}$ can cause any term to vanish.

(d). The factor q^β or any of its derivatives up to and including the β^{th}, save one,
$$\frac{\partial^\beta q^\beta}{\partial z_i^\beta}, \qquad (i = 1, 2)$$
will cause all terms to vanish in which it may occur.

(e). The factor r^γ or any of its derivatives up to and including the $2\gamma^{\mathrm{th}}$, save one,
$$\frac{\partial^{2\gamma} r^\gamma}{\partial z_1^\gamma \partial z_2^\gamma}$$
will cause all terms to vanish in which it may occur.

(f). Hence, the λ^{th} derivative of a term, $p^\alpha q^\beta r^\gamma$, will vanish for the point, $0:0:1$, unless
$$\beta + 2\gamma \leqq \lambda.$$

43. Applying these principles to the curve (9), Art. 16, we see at once that every term has
$$\beta + 2\gamma \leqq 5.$$

Hence, the first five polars must vanish. But for $\lambda = 6$ there are two terms whose sixth derivatives do not vanish,
$$-\,2^2 p^9 r^3 \text{ and } p^8 q^2 r^2.$$

Determining from these the sixth polar, the product of the tangential quantics at the sextuple point 13 of (7) is found to be
$$z_1^4 z_2^2 - 2 z_1^3 z_2^3 + z_1^2 z_2^4. \qquad (9)$$

This is a perfect square, and is precisely the dihedron form f_2^2. [(9), Art. 38.]

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Therefore, *the curve* (7) *has three cusps at the point* 13, and hence the same also at the other three critical points.

Thus we have
$$C \sim f_2^2. \tag{10}$$

This may be called the *normal* correspondence between f_2^2 and a ternary form of degree 18, while that between f_2^2 and P^2 of degree 12 *exists only because* (4) breaks up into straight lines through the critical points.

44. It remains to discover the ternary form to which f_1^2 corresponds. From the identity (8), Art. 38, we derive
$$2f_1^2 = 2^3 f_3^3 - 2 \cdot 3^3 f_2^2. \tag{11}$$
From (3) and (10),
$$A^3 - 2 \cdot 3^3 C \sim 2^3 f_3^3 - 2 \cdot 3^3 f_2^2. \tag{12}$$

Calling the left of (12) C', we have
$$C' \sim 2f_1^2. \tag{13}$$

45. The above results may be made more general as follows:

THEOREM IV.*

If the ternary forms α, β *correspond to the binary forms* a, b *of degree* λ, μ, *then the product*
$$\alpha\beta \sim ab$$
and the sum
$$\alpha + \beta \sim a, \ b \ \text{ or } \ a + b,$$
according as
$$\lambda < \mu, \ > \mu \ \text{ or } \ = \mu.$$

The proof follows directly from the principles of higher plane curves, since the correspondence in question is determined by finding the *lowest non-vanishing polar of each curve with respect to the given point.*

By this theorem a more general form than C corresponds to f_2^2, namely,
$$C + \delta A P^2 \sim f_2^2, \tag{14}$$
where δ is an arbitrary constant.

Here the binary correspondent of C, which is of *lower degree* than that of AP^2 [Art. 39, (1)], *prevails also* for the composite form.

* The chief theorems are numbered consecutively. See Arts. 22, 23, 24.

Likewise C' [(12), (13)] may be generalized, for if we put

$$A^3 - 2\cdot 3^3 C + \delta A P^2 = C'',$$

we have
$$C'' \sim 2f_1^2. \tag{15}$$

46. THEOREM V.

No invariant form under G_{120} can correspond to any binary form which involves odd powers of the cubic forms f_1, f_2.

Proof: If an odd power of either f_1 or f_2 alone is involved, then the form is of odd degree. This is impossible, since the degree of the *binary* form must equal the order of multiplicity of the point in question, which is always $2(n+\mu)$ according to theorem III, Art. 24.

If an odd power of the product only, $f_1 f_2$, is involved, the form is then of even degree, but is *non-symmetric* in z_1, z_2 [(10), Art. 38]. This is impossible, since the *ternary* forms are symmetric functions of z_1, z_2, z_3. [Art. 20]. Hence, all their polars with respect to the coordinate vertex, $0:0:1$, through which all the invariant curves pass [Art. 24], are *symmetric* functions in z_1, z_2, and similar statements hold with reference to the other three fundamental points.

Therefore, *all invariant forms under G_{120} have as their binary correspondents at the point** 13, *rational, integral functions of*

$$f_1^2,\ f_2^2,\ f_3.$$

47. The complete enumeration of binary correspondents for all ternary forms of degree 6, 12 and 18 may now be given as follows:

$$m_1 A \sim 2m_1 f_3,$$

$$m_1 A^2 + m_2 P^2 \sim \begin{cases} 2^2 m_1 f_3^2, & \text{if } m_1 \neq 0, \\ m_2 f_2^2, & \text{if } m_1 = 0,\ m_2 \neq 0. \end{cases}$$

$$m_1 A^3 + m_2 C + m_3 A P^2 \sim \begin{cases} m_2 f_2^2, & \text{if } m_1 = 0,\ m_2 \neq 0; \\ 2^3 m_1 f_3^3, & \text{if } m_1 \neq 0,\ m_2 = 0; \\ 2m_3 f_2^2 f_3^2, & \text{if } m_1 = m_2 = 0,\ m_3 \neq 0; \\ 2m_1 f_1^2, & \text{if } m_2 = -2\cdot 3^3 m_1,\ \neq 0; \\ 2^3 m_1 f_3^3 + m_2 f_2^2, & \text{if } m_2 \neq -2\cdot 3^3 m_1 \neq 0. \end{cases}$$

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*The foregoing investigation, which has been given with respect to the point 13, applies equally well to all four critical points, since these are conjugate under $G_{24}^{(1)}$.

The simplest invariant forms independent of P^2 corresponding directly to the fundamental even binary forms are

$$\left\{\begin{array}{l} \tfrac{1}{2}A \sim f_3, \\ \quad C \sim f_2^2, \\ \tfrac{1}{2}C' \sim f_1^2. \end{array}\right\} \tag{16}$$

THE GENERAL REDUCTION THEOREM. Arts. 48–52.

48. *Definition.* An invariant form is said to be *reduced* if it does not contain P^2 as a factor.

LEMMA.

Two reduced invariant forms of the same degree, which have the same binary correspondent at one of the vertices, can differ only in such terms as contain P^2 as a factor.

Proof: Let R and R' be two reduced ternary forms, each of degree $6n$, having the same tangential quantic at the point 13. We are to prove

$$R - R' = P^{2\mu} \cdot R''_{6(n-2\mu)}, \tag{1}$$

where R'' contains no factor of P and is zero if $\mu = 0$.

Since R and R' are *reduced* forms, the common tangential quantic at the point $0:0:1$, is of degree $2n$ [Art. 24], and involves only z_1 and z_2 [Art. 46]. Hence, by principles of higher plane curves,

(a). R and R' each contain no terms of degree less than $2n$ in z_1 and z_2.

(b). They are *identical* in the terms of degree $2n$ in z_1 and z_2.

(c). And, therefore, they can differ only in terms of degree *higher* than $2n$ in z_1 and z_2.

Now, $R - R'$ is an invariant form of degree $6n$, since R and R' are such by hypothesis. Then $R - R'$ must have at 13 a tangential quantic of degree $2(n+\mu)$. [Art. 24.]

Two cases now arise:

(I). $\mu = 0$. The tangential quantic is then of degree $2n$ which, by (c), is impossible, since $R - R'$ contains no terms of degree $\leqq 2n$ in z_1 and z_2. Hence,

$$R - R' = 0. \tag{2}$$

(II). $\mu \neq 0$. This means that $R - R'$ actually contains $P^{2\mu}$ as a factor, *in*

which case alone [Art. 41] can a form have a tangential quantic of degree higher than $2n$. Hence, the conclusion (I) is established.

Corollary.

49. If one of two ternary forms of the same degree has the factor $P^{2\mu}$ $(\mu \geqq 1)$, while the other does not, the degree of the binary correspondent of the former is greater by 2μ than that of the latter.

THEOREM VI.

50. *The most general invariant form under* G_{120} *is a rational integral function of the forms* A, P^2, C.

Proof: Let R_1 be the most general invariant form of degree $6n$, and let Q_1 be the resulting *reduced* form of degree $6(n - 2\mu_1)$.

Suppose the binary correspondent of Q_1 is given by

$$Q_1 \sim G_1(f_3, f_2^2, f_1^2), \tag{1}$$

where G_1 is a rational integral function of the *even* binary forms of degree $2(n - 2\mu_1)$ in z_1, z_2 [Arts. 24, 46.]

Now, a known invariant form of degree $6(n - 2\mu_1)$ in z_1, z_2, z_3 can be constructed, having G_1 for its binary correspondent,

$$S_1 \equiv G_1(\tfrac{1}{2}A, C, \tfrac{1}{2}C'). \tag{2}$$

For the arguments in (1) are weighted 2, 6, 6 respectively in the variables, and they are the binary forms corresponding *directly* to the arguments in (2), which are weighted 6, 18, 18 respectively. See (16), Art. 47.

Therefore, Q_1 and S_1 are two *reduced* forms having the same binary correspondent, which, by the lemma [Art. 48] are then either identical or differ only in terms involving P^2.

Thus, either $$Q_1 - S_1 \equiv 0, \tag{3}$$

in which case the theorem is proved, or

$$Q_1 - S_1 = R_2, \tag{4}$$

where R_2 is divisible by $P^{2\mu_2}$ $(\mu_2 \geqq 1)$.

Call the quotient Q_2 of degree $6(n - 2\mu_1 - 2\mu_2)$ and suppose

$$Q_2 \sim G_2(f_3,\ f_2^2,\ f_1^2). \tag{5}$$

Then, as before, set up the known form

$$S_2 \equiv G_2(\tfrac{1}{2}A,\ C,\ \tfrac{1}{2}C') \tag{6}$$

of degree $6(n - 2\mu_1 - 2\mu_2)$ and having the same binary correspondent as Q_2.

Hence, either $$Q_2 - S_2 = 0, \tag{7}$$

in which case $$R_2 = P^{2\mu_2} G_2$$

and $$R_1 = P^{2(\mu_2+\mu_1)} G_2 + P^{2\mu_1} G_1$$
$$= \phi(P^2,\ A,\ C), \tag{8}$$

or else $$Q_2 - S_2 = R_3, \tag{9}$$

where R_3 is divisible by $P^{2\mu_3}$ $(\mu_3 \geqq 1)$.

If this process be continued, we must reach. after a finite number of steps $\varkappa$, a form independent of P^2, since the finite degree $6n$ is successively reduced by multiples of 6, the divisor being always a power of P^2.

Hence, *after* the $\varkappa^{\text{th}}$ reduction, we shall have

$$Q_\kappa - S_\kappa = 0 \tag{10}$$

and $$Q_\kappa = S_\kappa = G_\kappa(\tfrac{1}{2}A,\ C,\ \tfrac{1}{2}C'), \tag{11}$$

where G_κ is a function of degree $6[n - 2(\mu_1 + \mu_2 + \ldots + \mu_\kappa)]$ in z_1, z_2, z_3. So that

$$R_\kappa = P^{2\mu_\kappa} G_\kappa. \tag{12}$$

But $$Q_{\kappa-1} = R_\kappa + G_{\kappa-1} = P^{2\mu_\kappa} G_\kappa + G_{\kappa-1}.$$

Hence $$R_{\kappa-1} = P^{2(\mu_\kappa+\mu_{\kappa-1})} G_\kappa + P^{2\mu_{\kappa-1}} G_{\kappa-1}.$$

Likewise,

$$R_{\kappa-2} = P^{2(\mu_\kappa+\mu_{\kappa-1}+\mu_{\kappa-2})} G_\kappa + P^{2(\mu_{\kappa-1}+\mu_{\kappa-2})} G_{\kappa-1} + P^{2\mu_{\kappa-2}} G_{\kappa-2},$$

$$R_{\kappa-3} = P^{2(\mu_\kappa+\mu_{\kappa-1}+\mu_{\kappa-2}+\mu_{\kappa-3})} G_\kappa + P^{2(\mu_{\kappa-1}+\mu_{\kappa-2}+\mu_{\kappa-3})} G_{\kappa-1} + P^{2(\mu_{\kappa-2}+\mu_{\kappa-3})} G_{\kappa-2} + P^{2\mu_{\kappa-3}} G_{\kappa-3},$$

. . . .

. . . .

$$R_1 = P^{2(\mu_\kappa+\mu_{\kappa-1}+\ldots\mu_1)} G_\kappa + P^{2(\mu_{\kappa-1}+\mu_{\kappa-2}+\ldots\mu_1)} G_{\kappa-1} + \ldots\ P^{2\mu_1} G_1.$$

Therefore, we have

$$\begin{aligned} R_1 &= \phi_1(P^2, G_1, G_2, \ldots G_\kappa) \\ &= \phi_2(P^2, A, C, C') \\ &= \phi\ (P^2, A, C), \end{aligned} \tag{13}$$

where the ϕ's are rational, integral functions.

51. Hence, *every absolute invariant form under G_{120}, throwing off the required factor in z_1, z_2, z_3, is a rational integral function of the fundamental forms A, P^2, C.*

Out of these forms must be constructed the invariant fractions, which return absolutely to themselves after canceling the common factor in numerator and denominator thrown off under any quadratic transformation of the group.

Since A is the only form of degree 6, there is no fraction of that order. The simplest fractions of degree 12, 18 and 24 respectively are

$$\frac{A^2}{P^2};\ \frac{A^3}{C},\ \frac{AP^2}{C};\ \frac{A^4}{P^4},\ \frac{AC}{P^4}.$$

52. It was shown [Art. 17] that the form P is the only *fundamental relative invariant form under $G_{24}^{(1)}$*, and that it throws off the factor (-1). It also throws off (-1) under the generator T' in addition to the factor r^2 in the z's required by the theorem of Art. 22.

It is, therefore, the *fundamental relative invariant form under G_{120}*, and all other relative invariants are formed from the product

$$P^{2\kappa+1} \cdot \phi\,(A, P^2, C).$$

However, P is an absolute invariant under the alternating group G_{60}, and the complete form-system for this subgroup is, therefore,

$$A,\ P,\ C.$$

There is then for G_{60} an absolute invariant fraction of degree 6,

$$\frac{A}{P}.$$

There is no form of degree 12 in addition to those given in Art. 51

For the degrees 18 and 24 the following non-reducible forms for G_{60} are to be added:

$$\frac{A^3}{P^3}, \quad \frac{C}{P^3}, \quad \frac{A^2P}{C}; \quad \frac{A^4}{PC}.$$

It will be seen that these fractions are all of type (III) or (IV), Art. 19, in which forms all relative invariant fractions under G_{120} must occur.

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NOTE.—In Part First, the last foot-note to Art. 1 should read, pp. 279–291, and the last foot-note to Art. 3 should read, p. 283. The heading above Art. 17 should read, Arts. 17–20.